where y is the lateral deflection, EI is the bending stiffness, and P is the tangential follower column load at the tip.

The tip boundary conditions appropriate to the follower force are

$$\text{at} \qquad x=\ell: \; EI\frac{d^2y}{dx^2}=0; \; EI\frac{d^3y}{dx^3}=F_t \tag{3}$$

and the root boundary conditions are

$$\text{at} \qquad x=0: \; y=0; \; \frac{dy}{dx}=0 \tag{4}$$

The solution is

$$y=F_t/k^3EI[\cos k\ell \sin kx - \sin k\ell \cos kx + \sin k\ell - kx\cos k\ell] \tag{5}$$

where $k^2=P/EI$.

For the deflection at the tip, y_t, this becomes

$$y_t=(F_t/k^3EI)(\sin k\ell - k\ell\cos k\ell) \tag{6}$$

The effective spring constant or flexibility, α_t, which enters into the equation of motion for the tip mass as a single-degree-of-freedom system is

$$\alpha_t = y_t/F_t = (\sin k\ell - k\ell\cos k\ell)/k^3EI \tag{7}$$

This formulation as a single-degree-of-freedom system is exactly that of previous authors (e.g., Ref. 2, p. 11) and corresponds to El Naschie's analysis as well

Examination of Eq. (1) for the spring constant discloses that with increasing follower force load on the column (i.e., increasing k) the spring constant α_t decreases, becoming equal to zero when

$$\sin k\ell = k\ell\cos k\ell \tag{8}$$

The column is effectively infinitely stiff to loads at the tip of this value of k and thus the frequency of the system comprising the massless column and the tip mass theoretically approaches infinity as the condition $\tan k\ell = k\ell$ is approached. Above this value of $k\ell$ the spring constant α_t is negative, representing a nonoscillatory divergent type of instability.

The spring constant for a transverse force F_a applied to a cantilever column at a point other than its tip ($x=a<\ell$), with a tangential follower load at the tip, is easily obtained by noting that, between the point of application of the transverse force and the tip, integration of Eq. (2) shows that the column has no curvature. Thus the slopes at the point of application of the transverse load and at the tip are identical and the deflection at the point of application of the transverse force is given by

$$y_a=(F_a/k^3EI)(\sin ka - ka\cos ka) \tag{9}$$

The spring constant at point a is then

$$\alpha_a=(1/k^3EI)(\sin ka - ka\cos ka) \tag{10}$$

Clearly for a massless column with a concentrated mass at a only, the value of k for which α_a vanishes varies with a, which unequivocally shows that the instability is dynamic, being dependent on the mass distribution.

To emphasize the extent to which the results obtained for the idealized massless structure with tip mass under tangential follower force loadings are special and nonrepresentative of the behavior found by analysis of more realistic models of columns under tangential follower force loading, Bolotin[2] has presented the results for a massless column with two concentrated masses at different locations, in which case there are two natural frequencies which tend to coalesce with increasing follower force load leading to oscillatory divergent instability. In a similar vein, Panovko and Gubanova[4] have shown that if the rotational inertia of the tip mass is taken into account, the resulting two frequencies also tend to coalesce with increasing follower force load and give rise to oscillatory divergent instability.

Finally, it may be noted (as El Naschie has in an earlier paper cited in Ref. 1) that Eq. (1) gives precisely the classical Euler buckling load (i.e., with conservative compression forces) of the column with one end fixed and the other pinned.[6] This follows from the fact that Eqs. (7) and (8) correspond to the condition that $y=0$ at $x=\ell$ regardless of the applied force F_t. Thus, since the other boundary conditions, Eqs. (3) and (4), are those of a fixed-pinned column, the condition $y_t=0$ leads to the classical Euler buckling load. However, it would be erroneous to assume from this fact that there is any physical similarity between the classical buckling of a fixed-pinned column and the instability of a cantilever column under tangential follower forces. The difference is obvious if Eq. (6), which indicates diminishing flexibility to transverse tip loading of a column under tangential follower forces as the latter force is increased toward the critical load, is compared with the well-known increasing flexibility to transverse loading of the classical (conservatively loaded) column as the compressive force is increased toward the classical buckling load.

Thus, although in the special case of an ideal massless structure with a single concentrated mass under tangential follower forces, the onset of instability is governed by the reversal of sign of the stiffness against transverse loads and is hence predictable from static criteria, the nature of the instability is essentially dynamic and there is no analogy to the classical Euler buckling problem. Several evidences of the dynamic nature of the problem have been indicated in the foregoing discussion, including the sensitivity of the instability load to the location of the single concentrated mass, to the addition of other concentrated masses or to the inclusion in the analysis of the distributed mass of the column structure itself.

References

[1] El Naschie, M. S., "Some Remarks on the Beck Problem," *AIAA Journal*, Vol. 15, Aug. 1977, pp. 1200-1201.

[2] Bolotin, V. V., *Nonconservative Problems of the Theory of Elastic Stability*, Macmillan Co., New York, 1963, pp. 7-18; 55-58; 90-94.

[3] Liepholz, H., *Stability Theory*, Academic Press, New York, 1970, pp. 212-213.

[4] Panovko, Y. G. and Gubanova, I. I., *Stability and Oscillation of Elastic Systems*, Consultants Bureau, New York, 1965, pp. 57-72.

[5] Von Kármán, T. and Biot, M. A., *Mathematical Methods in Engineering*, McGraw-Hill Book Co., New York, 1940, p. 224.

[6] Timoshenko, S. P. and Gere, J. M., *Theory of Elastic Stability*, McGraw-Hill Book Co., New York, 1961, p. 53.

Reply by Author to A. H. Flax

M. S. El Naschie*
University of Riyadh, Saudi Arabia

THE author is particularly glad that his small Note has triggered Flax's valuable comments and, in particular, his

Received Feb. 3, 1978.
Index category: Structural Stability.

*Assistant Professor.

criticism of the explanation given by Bolotin for the instability of columns having a tip mass under a tangential force and his own contribution to this problem.

Flax's analysis has removed many ambiguous points, for which I would like to thank him. I also agree with his views in general, but I differ in some points, which I would like to point out.

1) Although the closeness between the two values 20.05 and 20.19 seems to be a coincidence, the agreement between 20.19 and the buckling load of a fixed-pinned Euler column under conservation dead load is not insignificant.[1] In fact, it can be shown that this will hold true for a wide class of massless structures with a tip concentrated mass under follower forces. Flax' own analysis is a step in this direction. A good example for this is the well-known Leipholz problem. There are, of course, two Leipholz problems.[4] The first is the original problem[2] with distributed mass, where he found that the critical value is $P^c = 40.7\ EI/\ell^3$. The second is the corresponding massless problem with a tip concentrated mass. The critical load for this structure $P^c \cong 53.75\ EI/\ell^3$ was probably quoted for the first time by the author in Ref. 1. This value is also the critical value of a fixed-pinned Euler column. For this reason it seems, at least in some sense, to be a tautology to say that "in this special case the dynamical buckling is predictable from the static criteria," as stated by Flax, or "that a statical method could be used to find the buckling load of a nonconservative system," as stated by the author.

The author feels that it is misleading to speak of realistic models in the field of nonconservative follower forces. If we are ignoring effects like damping and nonlinearity, then we must regard the work done in this field as being mainly of theoretical interest. The "method" of transformed conservative statical system (T.C.S.S.) may be of some use in this sense.

3) It is interesting to see how a result thought to be well known and understood is still open to controversy, as in the present case. I think that many questions are still open in this respect, especially those concerning the postcritical behavior of nonconservative systems, and why, in the particular problem considered by the author, the conclusion of imperfection insensitivity agrees with the nonlinear dynamical analysis of Burgess,[3] which indicates soft flutter.

4) In conclusion, the author would like to make a few comments to the references cited by Flax. First, with regard to Ref. 3, it merely quotes the results of Pflüger without any analysis. The work of Pflüger does not deal explicitly with massless structures and is misleading in the interpretation that (m) must become infinity in order to represent a constraint leading to static buckling. In fact, any small mass different from zero would lead to the same result. However, having said that, the author painfully admits that he overlooked the undoubted fact that Bolotin obtained the critical value $P^c = 20.19\ EI/\ell^3$ long ago in his classic book. Nevertheless, in view of the deeper insight gained into this problem by the present renewed discussion, this mistake seems likely to have been a useful rather than a harmful one.

References

[1] El Naschie, M. S., "Post Critical Behavior of the Beck Problem," *Journal of Sound & Vibration*, Vol. 38, No. 3, 1976, pp. 341-344.

[2] Leipholz, H., "Die Knicklast des einseitig eingespamiten Stabes mit gleichmassig verteilten Tangential langsbelastung," *Zeitschrift für Angewandte Mathematik und Physik*, Vol. XIII, 1962, pp. 581-589.

[3] Burgess, I. W., "Ritz Method in Non-Conservative Instability Problems. A Simple Example," *International Journal of Mechanical Sciences*, Vol. 16, 1974, pp. 651-659.

[4] El Zaid, R. Z., "The Stability of Structural Systems under Polygenic Forces and their Numerical Treatment," B.Sc. Thesis, Civil Eng. Dept., University of Riyadh, Saudi Arabia, May 1978.

Comment on "A Finite-Element Approach for Nonlinear Panel Flutter"

G. Prathap*

Indian Institute of Technology, Madras, India

A USEFUL attempt to employ a finite-element method to study the nonlinear effects due to midplane stretching on panel flutter has been made by Mei.[1] However, the following reservations have to be made about the analysis in Ref. 1.

Following Mei,[1] for a two-dimensional flat plate of length a, thickness h, and mass per unit area m, the differential equation of motion is written as

$$D\frac{\partial^4 w}{\partial x^4} - (N_x + N_{x0})\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial t^2} = p \tag{1}$$

where

$$N_x = \frac{Eh}{2a}\int_0^a \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x \tag{2}$$

is the membrane force induced by large deflection, N_{x0} is the initial in-plane loading (tension positive), and $D = Eh^3/12(1-\nu^2)$ is the bending rigidity. The first observation regards the manner in which the axial loading N_{x0} is introduced. If it is obtained by an initial relative displacement between the two edges so that these edges remain immovable subsequently, the membrane force N_x is as given in Eq. (2) above. However, if N_{x0} is obtained by the action of an applied load at the edge, so that relative movement between the edges is possible (i.e., movable) during vibration, then $N_x = 0$, i.e., no membrane action due to stretching will occur within the first order nonlinear theory considered here.

The stiffness equation of motion for Eq. (1) in Ref. 1 is based on Refs. 2 and 3 and it contains an error in the evaluation of the nonlinear stiffness due to the membrane force. Note that Eq. (2) requires a constant membrane force N_x over the length of the plate $x=0$ to $x=a$, whereas Eq. (7) of Ref. 1 requires that N_x be a constant within an element of length l, depend only on the element modal displacements $\{u_e\}$, and therefore that it vary from element to element. This is obviously based on an erroneous assumption that for an element lying between $x=x'$ and $x=x'+l$, the stretching force is given by

$$N'_x = \frac{Eh}{2l}\int_{x'}^{x'+l} \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x \tag{3}$$

This describes an entirely different model from that described by Eq. (2). In other words, Eq. (2) describes a realistic model in which an in-plane longitudinal deformation is allowed, whereas Eq. (3) requires that no in-plane deformation be permitted and that each point on the panel have only a vertical deflection. The analysis in Ref. 1 can be improved by the author by redefining N_x so that it now takes into account the nature of Eq. (2) and therefore depends on the system nodal displacements $\{u\}$ and does not vary from element to element according to the element nodal displacements $\{u_e\}$ as in Eq. (7).

Received Feb. 14, 1978.

Index categories: Aeroelasticity and Hydroelasticity; Structural Dynamics; Supersonic and Hypersonic Flow.

*Research Assistant, Fibre Reinforced Plastics Research Centre.